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m-Particle Correlations in the N-Particle McKean Model

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This work extends the comparison between exact and approximate solutions of the McKean model to finite particle numbers. We derive the coupled linear equations of motion for the m-body densities (BBGKY hierarchy) and the corresponding nonlinear equations for the m-body correlation functions. We calculate the stable fixed points and the subspace admitting a probabilistic interpretation for both descriptions of the model. Neglecting higher correlations with $m > n$, we obtain approximate solutions, which are compared to the exact one. In this way various truncation effects can be studied, such as the appearance of saddle points and unphysical trajectories. Finally, we linearize the truncated equations for the correlations about the stable fixed point, and calculate the relaxation times up to $O(N^{-1})$.

KEY WORDS: BBGKY hierarchy; correlations; cluster expansion; truncation.

1. INTRODUCTION

A many-body system can be described by the N-body probability distribution function f_N which evolves in time according to the Liouville equation. By taking traces with respect to $N-m$ ($1 \leqslant m \leqslant N$) particles, these equations can be transformed into the BBGKY hierarchy of coupled equations which couple the m-body reduced distribution functions f_m to f_{m+1}. In the general case this hierarchy cannot be solved exactly, and some approximations must be made. By a cluster expansion of the distribution functions into one-particle distributions and correlations g_m, a new hierarchy for the $\{f_1, g_m \mid 2 \leqslant m \leqslant N\}$ is obtained that can be truncated by either neglecting or modeling of higher correlations g_m ($m > n$). Thus, the mean field theories are obtained by disregarding all correlations, while the Boltzmann collision

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integral is a model for g_2. On the next level one obtains equations of motion for g_2 in the polarization approximation $g_3 = 0$ and $g_2 < f_1 \cdot f_1$, which describe the time evolution of the two-particle correlations for weak coupling.[1] In the strong coupling, time-independent case the Kirkwood approximation for g_3 has been used to derive the hypernetted-chain (HNC) and related integral equations for the static two-particle correlations.[2]

Clearly any approximation schemes must not violate basic physical principles such as the conservation laws[3] and the trace relations for the reduced distribution functions.[4] Here we want to address two problems which may arise during the time evolution of a system. First, in the derivation of kinetic equations the Bogoliubov assumption is usually invoked, i.e., truncations are justified by assuming that any initial correlations are quickly attenuated. Second, approximate evolution equations do not necessarily guarantee that the system remains for all times in the subspace where a probabilistic interpretation is possible. Because of the H-theorem, this cannot happen on the level of the Boltzmann equation for f_1, but there seems to be no *a priori* reason why such an unphysical behavior could not occur if the truncation is made on a deeper level. This can in fact be explicitly demonstrated for the first two equations of the BBGKY hierarchy.

Here we do not give a general answer to these questions. Instead we want to use the McKean model[5] as a test case. It has been shown previously that this model is in fact exactly solvable for $N \to \infty$.[6] In the present paper we want to employ the McKean model with finite N in order to study the Bogoliubov assumption and possible violations of the probabilistic interpretation during the time evolution.

The paper is organized as follows. As a motivation for our work we critically discuss in Section 2 a derivation of an H-theorem for nonideal gases.[1] We derive an expression for the time evolution of the entropy and show in particular that the trace relations must be applied *before* taking the thermodynamic limit in order to obtain physically reasonable results. In the subsequent discussion of the McKean model we use the same kind of argument in a different direction: We use an extended H-theorem for the McKean[7] model in order to identify the physically allowed subspace admitting a probabilistic interpretation during the entire time evolution of the system. For that purpose we introduce in Sections 3 and 4 the McKean model and deduce the BBGKY hierarchy in closed form as well as the explicit first equations of the hierarchy for the correlation functions. It is advantageous to discuss both treatments in parallel. We determine the equilibrium subspace and the physically (p-) allowed region admitting a probabilistic interpretation in Sections 5 and 6. Next we show in Section 7 that truncation leads to unphysical trajectories. In contrast to the case of an infinite number of particles, $N \to \infty$, these problems cannot be overcome by regarding only trajectories starting in the p-allowed region.[6] A further restriction on the admissible states of the system is needed. For this purpose we consider in Section 8 the sign of the entropy production. Finally, in Section 9, we linearize the equations for the correlations about the stable fixed point and obtain closed formulas for the eigenvalues up to order $1/N$. It turns out that the higher correlations contain slowly relaxing components with amplitude $O(1/N)$.

2. CORRELATIONS, TRACE RELATIONS, THERMODYNAMIC LIMIT, AND ENTROPY IN THE CLASSICAL KINETIC THEORY

We want to study the time evolution of the entropy in the classical kinetic theory, taking the two-particle correlations into account. The Liouville equation for the classical N-particle probability density $f_N(x_1,..., x_N, t)$ with the phase space coordinates $x_i = (\mathbf{r}_i, \mathbf{p}_i)$, $i = 1,..., N$, is given by

$$\frac{\partial f_N}{\partial t} = \sum_i \left(\frac{\partial H}{\partial \mathbf{r}_i} \cdot \frac{\partial f_N}{\partial \mathbf{p}_i} - \frac{\partial H}{\partial \mathbf{p}_i} \cdot \frac{\partial f_N}{\partial \mathbf{r}_i} \right) \tag{2.1}$$

where H is the Hamilton function involving the central two-body interaction ϕ_{ij}. We normalize according to

$$\int dx_1 \cdots dx_N f_N(x_1,..., x_N, t) = 1 \tag{2.2}$$

and introduce reduced m-particle distribution functions by

$$f_m(x_1,..., x_m, t) = V^m \int f_N(x_1,..., x_N, t)\, dx_{m+1} \cdots dx_N \tag{2.3}$$

where V is the volume of the system. Integrating with respect to $N - m$ variables, one obtains for $m = 1, 2$ the first two equations of the BBGKY hierarchy,[1]

$$\frac{\partial f_1}{\partial t} = \frac{N-1}{V} \int \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} \cdot \frac{\partial f_2}{\partial \mathbf{p}_1}\, dx_2 \tag{2.4}$$

and

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2 \frac{\partial}{\partial \mathbf{r}_2} - \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial \phi_{12}}{\partial \mathbf{r}_2} \cdot \frac{\partial}{\partial \mathbf{p}_2} \right) f_2$$
$$= \frac{N-2}{V} \int \left(\frac{\partial \phi_{13}}{\partial \mathbf{r}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} + \frac{\partial \phi_{23}}{\partial \mathbf{r}_2} \frac{\partial}{\partial \mathbf{p}_2} \right) f_3\, dx_3 \tag{2.5}$$

Here $\mathbf{v} = \mathbf{p}/m$ is the velocity of the particles and it has been assumed that the system is translation invariant. In anticipation of later problems, we do *not* take the thermodynamic limit at this stage. We use now the cluster expansions

$$\begin{aligned} f_2(x_1, x_2) &= f_1(x_1)\, f_1(x_2) + g_2(x_1, x_2) \\ f_3(x_1, x_2, x_3) &= f_1(x_1)\, f_1(x_2)\, f_1(x_3) + f_1(x_1)\, g_2(x_2, x_3) \\ &\quad + f_1(x_2)\, g_2(x_1, x_3) + f_1(x_3)\, g_2(x_1, x_2) + g_3(x_1, x_2, x_3) \end{aligned} \tag{2.6}$$

with the correlation functions $g_m(x_1,..., x_m)$. These fulfill the trace relations

$$\int g_m(x_1,..., x_m)\, dx_i = 0, \qquad \forall i = 1,..., m \tag{2.7}$$

because of the definition (2.3) of the reduced distribution functions. Insertion of the cluster expansions (2.6) into Eqs. (2.4) and (2.5), dropping the three-particle correlations, and using the translational invariance yields

$$\frac{\partial f_1}{\partial t} = \frac{N-1}{V} \int \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} \cdot \frac{\partial g_2}{\partial \mathbf{p}_1}\, dx_2 \tag{2.8}$$

and

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \mathbf{v}_1 \frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2 \frac{\partial}{\partial \mathbf{r}_2} - \frac{\partial \phi_{12}}{\partial \mathbf{r}_1} \frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial \phi_{12}}{\partial \mathbf{r}_2} \frac{\partial}{\partial \mathbf{p}_2}\right) f_2 \\ &\quad = \frac{N-2}{N-1} \frac{\partial}{\partial t} (f_1(x_1)\, f_1(x_2)) \end{aligned} \tag{2.9}$$

The entropy density of the system is given by

$$S = -k_{\mathrm{B}} \frac{N}{V^2} \int f_1 \ln f_1\, dx_1 - \frac{1}{2} k_{\mathrm{B}} \frac{N(N-1)}{V^3} \int f_2 \ln \frac{f_2}{f_1 f_1}\, dx_1\, dx_2 \tag{2.10}$$

where the second term accounts for pairwise correlations.[1] Taking the derivative with respect to time and using Eqs. (2.4) and (2.5), one obtains

$$\begin{aligned} \frac{\partial S}{\partial t} &= \frac{\partial S_1}{\partial t} + \frac{\partial S_2}{\partial t} \\ &= -k_{\mathrm{B}} \frac{N}{V^2} \int \frac{\partial f_1}{\partial t} \ln f_1\, dx_1 - \frac{1}{2} k_{\mathrm{B}} \frac{N(N-1)}{V^3} \int \frac{\partial f_2}{\partial t} \ln \frac{f_2}{f_1 f_1}\, dx_1\, dx_2 \end{aligned} \tag{2.11}$$

In the correlation term $\partial S_2/\partial t$ we subtract a multiple of an integral which is zero to the order considered. For that purpose we follow ref. 1 and multiply the modified Eq. (2.8),

$$\begin{aligned}\frac{\partial f_1(x_1)\, f_1(x_2)}{\partial t} &= \frac{N-1}{V}\left(f_1(x_2)\int \frac{\partial \phi_{13}}{\partial \mathbf{r}_1}\cdot\frac{\partial g_2(x_1, x_3)}{\partial \mathbf{p}_1}\, dx_3 + f_1(x_1)\right.\\ &\quad\left.\times\int \frac{\partial \phi_{23}}{\partial \mathbf{r}_2}\cdot\frac{\partial g_2(x_2, x_3)}{\partial \mathbf{p}_2}\, dx_3\right)\end{aligned} \tag{2.12}$$

with

$$-\frac{1}{2} k_B V^{-2}\left(\frac{N}{V}\ln f_1 f_1 + \frac{N^2}{V}\ln\frac{f_2}{f_1 f_1}\right) dx_1\, dx_2 \tag{2.13}$$

integrate with respect to x_1 and x_2, and drop the third-order terms on the right-hand side. This yields

$$\begin{aligned}&-k_B\frac{N}{V^2}\int\frac{\partial f_1}{\partial t}\ln f_1\, dx_1 - \frac{1}{2}k_B\frac{N^2}{V^3}\int\frac{\partial f_1 f_1}{\partial t}\ln\frac{f_2}{f_1 f_1}\, dx_1\, dx_2\\ &\quad = -k_B\frac{N(N-1)}{V^3}\int\frac{\partial\phi_{12}}{\partial \mathbf{r}_1}\cdot\frac{\partial g_2(x_1, x_2)}{\partial \mathbf{p}_1}\ln f_1(x_1)\, dx_1\, dx_2\end{aligned} \tag{2.14}$$

Because of Eq. (2.8), one has then

$$\int\frac{\partial f_1 f_1}{\partial t}\ln\frac{f_2}{f_1 f_1}\, dx_1\, dx_2 = 0 \tag{2.15}$$

and the correlation term of the entropy can be written as

$$\begin{aligned}&\int\frac{\partial f_2}{\partial t}\ln\frac{f_2}{f_1 f_1}\, dx_1\, dx_2\\ &\quad = \int \partial\left(f_2 - \frac{N-2}{N-1} f_1 f_1\right)\Big/\partial t\, \ln f_2\, dx_1\, dx_2\\ &\qquad - \int \partial\left(f_2 - \frac{N-2}{N-1} f_1 f_1\right)\Big/\partial t\, \ln f_1 f_1\, dx_1\, dx_2\end{aligned} \tag{2.16}$$

The first term on the right-hand side is zero; this follows by applying Gauss' theorem to Eq. (2.9). The remaining term is evaluated with the help of Eq. (2.3). The correlation contribution to the time evolution of the entropy is then

$$\begin{aligned}\frac{\partial S_2}{\partial t} &\equiv \frac{1}{2} k_B \frac{N(N-1)}{V^3} \int \frac{\partial f_2}{\partial t} \ln \frac{f_2}{f_1 f_1} dx_1\, dx_2 \\ &= \frac{1}{2} k_B \frac{N(N-1)}{V^3} 2V \left(1 - \frac{N-2}{N-1}\right) \int \frac{\partial f_1}{\partial t} \ln f_1\, dx_1 \\ &= k_B \frac{N}{V^2} \int \frac{\partial f_1}{\partial t} \ln f_1\, dx_1 \\ &= -\frac{\partial S_1}{\partial t} \end{aligned} \tag{2.17}$$

which cancels the first-order contribution. The result $\partial S_2/\partial t = 0$ in Section 14 of ref. 1 is obtained by taking the thermodynamic limit in the second term on the right hand side of Eq. (2.16) in an incorrect manner,

$$\begin{aligned} &\lim_{N\to\infty} \int \partial \left(f_2 - \frac{N-2}{N-1} f_1 f_1\right) \Big/ \partial t \ln f_1 f_1\, dx_1\, dx_2 \\ &\quad \neq \int \lim_{N\to\infty} \partial \left(f_2 - \frac{N-2}{N-1} f_1 f_1\right) \Big/ \partial t \ln f_1 f_1\, dx_1\, dx_2 \\ &\quad = \int \frac{\partial g_2}{\partial t} \ln f_1 f_1\, dx_1\, dx_2 = 0 \end{aligned} \tag{2.18}$$

Here the last equality sign follows from the trace relation (2.7). One may therefore not drop the three-particle terms altogether in order to study the time evolution of the entropy $S = S_1 + S_2$ of Eq. (2.10). While Boltzmann's H-theorem $\partial S_1/\partial t \geqslant 0$ can be obtained by modeling g_2 with the help of Bogoliubov's assumption of complete weakening of initial correlations,[1] one also needs now models on the deeper level for g_2 (or f_3). This may lead to problems with the probability interpretation of the theory if the resulting equations of motion for f_1 and f_2 become incompatible because of a violation of the trace relations (2.3) as occurs, for example, in the Kirkwood superposition assumption

$$f_3(x_1, x_2, x_3) = \frac{f_2(x_1, x_2)\, f_2(x_1, x_3)\, f_2(x_2, x_3)}{f_1(x_1)\, f_1(x_2)\, f_1(x_3)} \tag{2.19}$$

In the following we will pursue such questions in a somewhat different direction. We investigate the McKean model, which is, in contrast to kinetic theory, not time-reversal invariant on the microscopic level. The physically allowed regions for the time evolution of correlation functions will then be studied with the help of an extended H-theorem.[7]

3. THE MCKEAN MODEL

We consider a system of N particles, $i = 1,\ldots, N$, where each particle can occupy two states, $e_i = \pm 1$ or $(\uparrow, \downarrow)$. After an interaction, two particles i, j will be found in states e_i^*, e_j^* according to

$$\left\{\begin{array}{l} e_i^* = e_j \\ e_j^* = e_i e_j \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} e_i^* = e_i e_j \\ e_j^* = e_j \end{array}\right\} \tag{3.1}$$

or, more explicitly,

$$\begin{array}{ll} +1, +1 & \begin{array}{l} \nearrow +1, +1 \\ \searrow +1, +1 \end{array} \\[2ex] -1\ +1 & \begin{array}{l} \nearrow -1, -1 \\ \searrow -1, +1 \end{array} \\[2ex] +1, -1 & \begin{array}{l} \nearrow +1, -1 \\ \searrow -1, -1 \end{array} \\[2ex] -1, -1 & \begin{array}{l} \nearrow -1, +1 \\ \searrow +1, -1 \end{array} \end{array}$$

both possibilities having the probability 1/2.

This scattering law is not time-reversal invariant. We describe the system by distribution functions that are symmetric in their arguments,

$$\begin{aligned} &f_N(e_i,\ldots, e_i,\ldots, e_j,\ldots, e_N, t) \\ &\quad = f_N(e_i,\ldots, e_j,\ldots, e_i,\ldots, e_N, t) \end{aligned} \tag{3.2}$$

and are normalized according to

$$\sum_{\substack{e_i = \pm 1 \\ 1 \leqslant i \leqslant N}} f_N(e_1,\ldots, e_N, t) = 1 \tag{3.3}$$

Similarly, we can define reduced m-particle distribution functions by

$$f_m(e_1,\ldots, e_m, t) = \sum_{\substack{e_i = \pm 1 \\ m < i \leqslant N}} f_N(e_1,\ldots, e_N, t) \tag{3.4}$$

A kinetic equation for the N-particle distribution function is simply given by the difference of distributions before and after the interaction assuming a transition rate $2/N$,

$$\begin{aligned}
&\frac{\partial}{\partial t} f_N(e_1,\ldots, e_N, t)\\
&\quad = \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} [f_N(e_1,\ldots, e_i,\ldots, e_j e_i,\ldots, e_n, t)\\
&\qquad + f_N(e_1,\ldots, e_i e_j,\ldots, e_j,\ldots, e_N, t) - 2f_N(e_1,\ldots, e_N, t)] \qquad (3.5)
\end{aligned}$$

As a consequence of the interaction law (3.1), a distribution function with all its arguments in state $+1$ is a constant of motion. We will show in Section 3 that this will lead to a continuum of equilibrium states. Since the distribution functions are symmetric in their variables, they differ only in the number of particles that are in state $+1$. Therefore we use, wherever possible, another notation:

f_m means that all m arguments have the value $+1$

$f_m(k\downarrow)$ means that exactly k arguments have the value -1

We obtain a hierarchy of evolution equations for the reduced distribution functions f_m by summing (3.5) over e_{m+1} to e_N. This yields

$$\begin{aligned}
&\sum_{\substack{e_i = \pm 1\\ m < i \leqslant N}} \frac{\partial}{\partial t} f_N(e_1,\ldots, e_N)\\
&\quad = \frac{\partial}{\partial t} f_m(e_1,\ldots, e_m)\\
&\quad = \frac{1}{N} \sum_{i < j \leqslant m} [f_m(e_1,\ldots, e_i,\ldots, e_i e_j,\ldots, e_m)\\
&\qquad + f_m(e_1,\ldots, e_i e_j,\ldots, e_j,\ldots, e_m) - 2f_m(e_1,\ldots, e_m)]\\
&\qquad + \frac{N-m}{N} \sum_{e_{m+1} = \pm 1} \sum_{i \leqslant m} [f_{m+1}(e_1,\ldots, e_i,\ldots, e_i e_{m+1})\\
&\qquad + f_{m+1}(e_i,\ldots, e_i e_{m+1},\ldots, e_{m+1}) - 2f_{m+1}(e_1,\ldots, e_{m+1})] \qquad (3.6)
\end{aligned}$$

It is sufficient to study the evolution of the distributions f_m with all arguments having the value $+1$, because the distributions with arbitrary arguments can be obtained with the help of Eqs. (3.3) and (3.4). This yields

$$\frac{\partial}{\partial t} f_m = \left(\frac{m^2}{N} - m\right) [2f_m - 2f_{m+1} - f_{m+1}(\downarrow) - f_{m+1}(2\downarrow)]$$

$$= \left(\frac{m^2}{N} - m\right) (3f_m - f_{m-1} - 2f_{m+1}) \qquad 1 \leqslant m \leqslant N \tag{3.7}$$

These equations represent the BBGKY hierarchy of the McKean model. It is a system of coupled linear differential equations with constant coefficients. The solution can be obtained with the ansatz $f_m \propto \exp \lambda_m t$. Obviously $\lambda_N = 0$ and our calculations will show that all $\lambda_{m<N}$ are negative. The system will therefore always reach equilibrium.

4. EQUATIONS OF MOTION FOR THE CORRELATIONS

We derive equations for the correlations by making use of the cluster expansion.[8,9] In the present case, where all arguments e_i have been set equal to $+1$, we cannot distinguish, for example, between the two-particle correlations $g_2(e_1, e_2)$ and $g_2(e_2, e_3)$. This leads to the appearance of combinatorial coefficients in the expansion. The first three expansions are

$$\begin{aligned} f_2 &= f_1^2 + g_2 \\ f_3 &= f_1^3 + 3f_1 g_2 + g_3 \\ f_4 &= f_1^4 + 6f_1^2 g_2 + 4f_1 g_3 + g_4 + g_2^2 \end{aligned} \tag{4.1}$$

Substitution into the BBGKY hierarchy (3.7) yields the equations of motion for the correlations g_m, the first five of which are

$$\partial_t f_1 = \left(\frac{1}{N} - 1\right) (3f_1 - 1 - 2f_1^2 - 2g_2) \tag{4.2}$$

$$\partial_t g_2 = \frac{1}{N} (6f_1^2 - 4f_1^3 - 2f_1 - 20f_1 g_2 + 12g_2 - 8g_2) - 6g_2 + 4g_3 + 8f_1 g_2 \tag{4.3}$$

$$\begin{aligned} \partial_t g_3 = {} & \frac{1}{N} (36f_1 g_2 + 27g_3 - 36f_1^2 g_2 - 48f_1 g_3 - 18g_4 - 6g_2 - 48g_2^2) \\ & + 12f_1 g_3 - 9g_3 + 6g_4 + 12g_2^2 \end{aligned} \tag{4.4}$$

$$\begin{aligned} \partial_t g_4 = {} & \frac{1}{N} (72f_1 g_3 + 48g_4 + 72g_2^2 - 12g_3 - 88f_1 g_4 - 72f_1^2 g_3 \\ & - 32g_5 - 264g_2 g_3 - 144f_1 g_2^2) - 12g_4 + 16f_1 g_4 + 8g_5 + 48g_2 g_3 \end{aligned} \tag{4.5}$$

$$\partial_t g_5 = \frac{1}{N}(120 f_1 g_4 + 75 g_5 + 360 g_2 g_3 - 20 g_4 - 140 f_1 g_5 - 240 g_2^3 - 720 f_1 g_2 g_3$$
$$- 120 f_1^2 g_4 - 420 g_3^2 - 560 g_2 g_4 - 50 g_6) - 15 g_5$$
$$+ 20 f_1 g_5 + 60 g_3^2 + 80 g_2 g_4 + 10 g_6 \tag{4.6}$$

For $m = N$ the coefficient multiplying g_{m+1} vanishes, so that the complete set of equations is closed in a natural manner, similar to the original BBGKY hierarchy (3.7). The structure of these equations for the correlations is more complicated, however, since they are nonlinear. They can only be integrated numerically.

5. EQUILIBRIUM STATES OF THE MODEL

We first take a look at the distribution functions. To obtain the fixed points we set $\dot{f}_m = 0$ in Eq. (3.7). As $f_N = \text{const}$, this leaves a linear system of $N-1$ nontrivial equations for the N equilibrium distributions. The explicit equilibrium solutions

$$f_m = \frac{(2^m - 1) f_1 - (2^{m-1} - 1)}{2^{m-1}} \tag{5.1}$$

span a one-dimensional subspace. There exists an N-tuple of functions $\{f_m \mid 1 \leqslant m \leqslant N\}$, where the $f_{m \geqslant 2}$ depend linearly on f_1. The equilibrium states are therefore lying on a straight line in the space of the $\{f_m\}$, which is shown as the dashed line in Fig. 1 for the case $N = 3$. They can be labeled

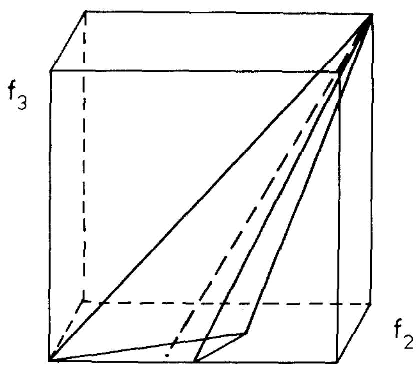

Fig. 1. The unit cube spanned by $\{f_1, f_2, f_3\}$. The p-allowed region forms the solid prism within the cube. The equilibrium states lie on the dashed line within the prism.

according to the value of f_1. With the help of Eq. (3.4), we can calculate the remaining distribution functions,

$$f_m(k\downarrow)=\frac{1-f_1}{2^{m-1}}, \qquad k\in\{1,\dots,m\} \tag{5.2}$$

Comparing Eqs. (5.2) and (5.1) shows that for $f_1=1/2$ one has a microcanonical ensemble, each state being occupied with the same probability.

Since Eqs. (4.2)–(4.6) are nonlinear, we cannot describe the equilibrium state in terms of the correlations as simply. Solving these equations, we also obtain a one-dimensional solution in form of a curve in the N-dimensional space of correlations. The equilibrium state is now described by a polynomial in $\{f_1, g_m\}$ which is implicitly given by

$$g_2=\frac{3}{2}f_1-\frac{1}{2}-f_1^2 \tag{5.3}$$

$$g_m=g_2\frac{dg_{m-1}}{df_1}, \qquad 3\leqslant m\leqslant 5 \tag{5.4}$$

The projections of this polynomial into the f_1–g_m planes are shown in Fig. 2.

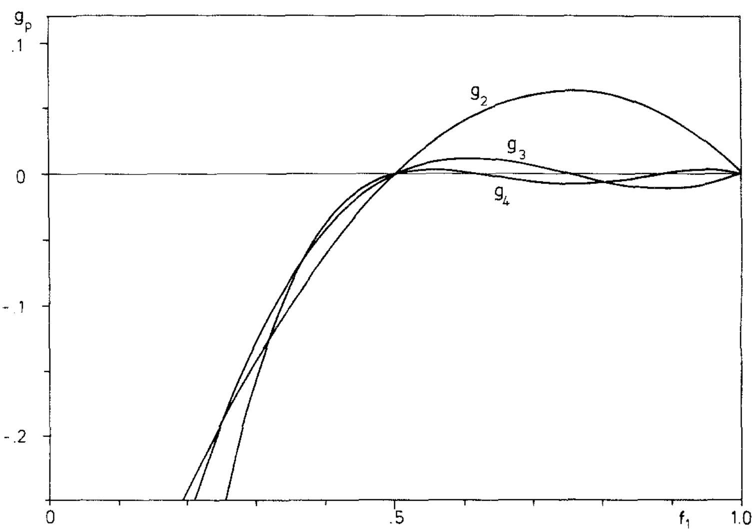

Fig. 2. Projection of the curve of fixed points in N-dimensional space onto each f_1–g_m plane for $m=2, 3, 4$.

6. PROBABILISTIC SUBSPACE

As a probability distribution each f_m must fulfill $0 \leqslant f_m \leqslant 1$, i.e., the $\{f_m\}$ lie in a hypercube. The $N+1$ distribution function $f_m(k\downarrow)$ for $0 \leqslant k \leqslant m$ and fixed m also can take only values from 0 to 1. This leads to a further restriction for the reduced distributions f_m, as can be seen by taking a closer look at the reduction (3.4),

$$f_m(k\downarrow) + f_m[(k+1)\downarrow] = f_{m-1}(k\downarrow), \qquad k \in \{0, 1,..., m-1\} \tag{6.1}$$

This gives m conditions and since all $f_m(k\downarrow)$ are nonnegative, the inequalities

$$f_m(k\downarrow) \leqslant f_{m-1}(k\downarrow) \tag{6.2}$$

must hold. These conditions can be expressed in terms of reduced distributions $\{f_m\}$, where all arguments are $+1$

$$(-1)^k f_m \leqslant \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} f_{m-1-k+j} \text{ with } k \in \{0, 1,..., m-1\} \tag{6.3}$$

This formula yields $N!$ conditions, since we have to vary m from 2 to N and also k from 0 to $m-1$. We regard, for instance, the case $N=2$, $m=2$:

$$\begin{aligned} k=0: &\qquad f_2 \leqslant f_1 \\ k=1: &\qquad f_2 \geqslant 2f_2 - 1 \end{aligned} \tag{6.4}$$

which describes a triangle in $\{f_1, f_2\}$ space. For $N=3$ we have, in addition, to consider $m=3$; this yields

$$\begin{aligned} k=0: &\qquad f_3 \leqslant f_2 \\ k=1: &\qquad f_3 \geqslant 2f_2 - f_1 \\ k=2: &\qquad f_3 \leqslant 1 - 3f_1 + 3f_2 \end{aligned} \tag{6.5}$$

Now the subspace of the unit cube spanned by $\{f_1, f_2, f_3\}$ in which all the relations (6.5) and (6.4) hold is the prism shown in Fig. 1 and the triangle corresponding to (6.4) alone is the projection of the prism on the $\{f_1, f_2\}$ plane. Similarly, an expansion to $N=4$ yields only further conditions between $f_4(k\downarrow)$ and $f_3(k\downarrow)$, so the prism of Fig. 1 is the projection of a higher dimensional prism to three dimensions, and so on. With the help of relations (6.3) we have thus explicitly constructed a subspace within the hypercube spanned by the $\{f_m\}$ in which a probabilistic interpretation of the theory is ensured. We will call that subspace the p-allowed region. The prisms in $\{f_m\}$ space translate into regions of rather complicated shape in the space of correlations $\{f_1, g_m\}$, because of the nonlinear equations (4.1).

7. TRUNCATION OF THE HIERARCHY FOR THE CORRELATIONS

In general the many-body problem cannot be solved exactly and one has to truncate the BBGKY hierarchy at some level. In this section, we investigate the consequences of a truncation of Eqs. (4.2)–(4.6) for the McKean model on the nth level by setting $g_{n+1}(t)=0$. The first n equations with $1 \leqslant m \leqslant n$ form a closed set. In the simplest approximation one neglects all two-particle correlations $g_2(t)=0$ and obtains

$$\partial_t f_1 = \left(\frac{1}{N} - 1\right)(3f_1 - 1 - 2f_1^2) \tag{7.1}$$

This equation has an attractive fixed point for $f_1 = 1/2$ and a repulsive one for $f_1 = 1$.[6] Its solution is

$$f_1(t) = \frac{1}{2}\left[1 + \frac{a e^{(1/N-1)t}}{1 - a(1 - e^{(1/N-1)t})}\right] \quad \text{with} \quad a = 2f_1(t=0) - 1 \tag{7.2}$$

i.e., the system reaches equilibrium faster for a larger number of particles N. On the next level $n=2$, we set $g_3 \equiv 0$ and the remaining equations are now

$$\partial_t f_1 = \left(\frac{1}{N} - 1\right)(3f_1 - 1 - 2f_1^2 - 2g_2) \tag{7.3}$$

$$\partial_t g_2 = \frac{1}{N}(6f_1^2 - 4f_1^3 - 2f_1 - 20f_1 g_2 + 12g_2) - 6g_2 + 8f_1 g_2 \tag{7.4}$$

These equations have in addition to the fixed points $(f_1, g_2) = (1/2, 0)$ and $(1, 0)$ another one at $(3/4, 1/16)$. We linearize Eqs. (7.3) and (7.4) about these fixed points, make an exponential ansatz $\propto \exp \lambda t$ for the solutions, and obtain the eigenvalues

$$\lambda = \frac{1}{2}\left[\left(3 - \frac{9}{N}\right) \pm \left(1 - \frac{30}{N} + \frac{65}{N^2}\right)^{1/2}\right] \quad \text{at } (1, 0) \tag{7.5}$$

$$\lambda = \frac{1}{2}\left[\left(\frac{3}{N} - 3\right) \pm \left(1 + \frac{6}{N} - \frac{7}{N^2}\right)^{1/2}\right] \quad \text{at } \left(\frac{1}{2}, 0\right) \tag{7.6}$$

$$\lambda = \frac{1}{2}\left[-\frac{3}{N} \pm \left(4 - \frac{12}{N} + \frac{17}{N^2}\right)^{1/2}\right] \quad \text{at } \left(\frac{3}{4}, \frac{1}{16}\right) \tag{7.7}$$

Inspection shows that the fixed point $(1/2, 0)$ is attractive for all particle numbers, while $(3/4, 1/16)$ is a saddle point and $(1, 0)$ is repulsive with complex eigenvalues for $2 < N < 28$.

In general, we have different sets of fixed points at each level n of truncation, which is a consequence of setting $g_{n+1}(t)=0$. We have seen in Section 5 that the exact equilibrium solution of the McKean model is a one-dimensional curve in the N-dimensional space $\{f_1, g_m\}$, which could also be labeled by choosing $g_{n+1}(t)=\text{const}$ as a parameter. Truncation in the nth level is achieved by setting this constant equal to zero. Therefore, the fixed points appearing at each truncation level n can be read off Fig. 2; they coincide with the zeros of g_{n+1}. In particular, the attractive fixed point $(1/2, 0)$ and the repulsive fixed point $(1, 0)$ occur at each level of truncation.

8. TIME EVOLUTION OF THE ENTROPY

The existence of saddle points poses a problem, since the truncated subdynamics may lead to unphysical trajectories: The case $N\to\infty$ is uncritical, as all saddle points already lie outside the p-allowed region for the correlations.[6] For finite N, however, some saddle points will lie in this region as it broadens, with a decreasing number of particles. In the case $N=3$, for example, the p-allowed region is sufficiently large to contain the saddle point $(3/4, 1/16)$. Trajectories which start in the p-allowed region outside the basin of the attractive fixed point $(1/2, 0)$, will eventually leave the p-allowed region during the course of the dynamic evolution of the system. The admissible initial conditions must therefore be restricted further. For that purpose we study the entropy of the McKean model, which is given by[7]

$$S=\tfrac{1}{2}\ln\Omega \tag{8.1}$$

with the Liapunov function

$$\Omega=\sum_{0\leqslant k\leqslant N}\binom{N}{k}[f_N(k\downarrow)]^2 \tag{8.2}$$

so that

$$\dot{S}=-\Omega^{-1}\sum_{0\leqslant k\leqslant N}\binom{N}{k}f_N(k\downarrow)\,\dot{f}_N(k\downarrow) \tag{8.3}$$

For $N=3$ one has explicitly

$$\dot{S}=-\Omega^{-1}[f_3\dot{f}_3+3f_3(\downarrow)\,\dot{f}_3(\downarrow)+3f_3(2\downarrow)\,\dot{f}_3(2\downarrow)+f_3(3\downarrow)\,\dot{f}_3(3\downarrow)] \tag{8.4}$$

Next, we expand the three-particle distribution functions into clusters and neglect the three-particle correlations g_3. The sign of the entropy production is

$$\begin{aligned}\operatorname{sgn} \dot{S} = -\operatorname{sgn}\{&(f_1^3 + 3f_1 g_2)(3f_1^2 \dot{f}_1 + 3\dot{f}_1 g_2 + 3f_1 \dot{g}_2)\\ &+ 3[f_1^2(1-f_1) - (3f_1 - 1) g_2]\\ &\times [2f_1 \dot{f}_1(1-f_1) - f_1^2 \dot{f}_1 - 3\dot{f}_1 g_2 - 3f_1 \dot{g}_2 + \dot{g}_2]\\ &+ 3(f_1 + f_1^3 - 2f_1^2 + 3f_1 g_2 - 2g_2)\\ &\times (\dot{f}_1 + 3\dot{f}_1 f_1^2 - 4\dot{f}_1 f_1 + 3\dot{f}_1 g_2 + 3f_1 \dot{g}_2 - 2\dot{g}_2)\\ &+ (1 - 3f_1 + 3f_1^2 - f_1^3 + 3g_2 - 3f_1 g_2)\\ &\times [-3\dot{f}_1(1-f_1)^2 + 3\dot{g}_2 - 3f_1 \dot{g}_2 - 3\dot{f}_1 g_2]\}\end{aligned} \tag{8.5}$$

Substituting $\dot{f}_1$ and $\dot{g}_2$ from hierarchy equations (4.2) and (4.3), one finds that this equation yields a curve in the (f_1, g_2) plane that separates a region $\dot{S} > 0$ which is allowed with respect to the entropy production from a forbidden region with $\dot{S} < 0$. In Fig. 3 this S-allowed region lies left of the dashed curve, while the p-allowed region lies within the solid curves. We show that the saddle point lies on the dashed curve and that it is not an isolated point with $\dot{S} = 0$ [such as (1/2, 0)]. For that purpose we prove

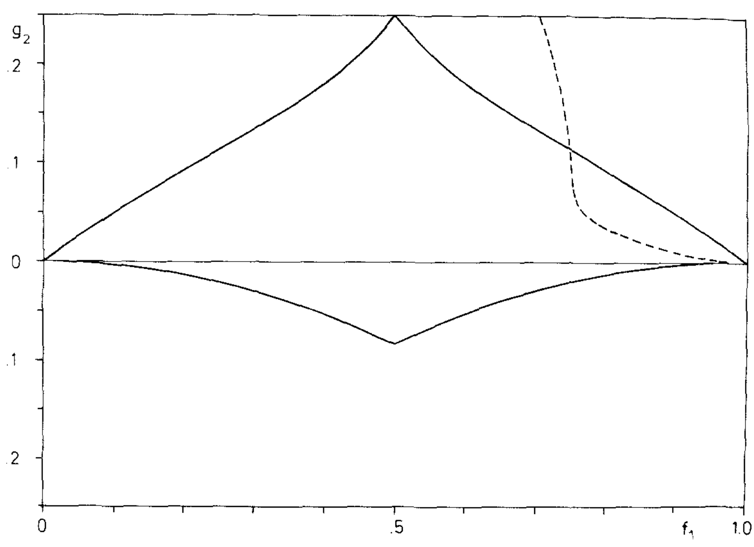

Fig. 3. The p-allowed (diamondlike area within the solid lines) and the s-allowed (left of the dashed line) regions for a truncated $[g_3(t) \equiv 0]$ system of three particles.

that in every ε circle around (3/4, 1/16) there are points with $\dot{S}>0$ and $\dot{S}<0$. We consider the curve

$$g_2 = \tfrac{3}{2} f_1 - \tfrac{1}{2} - f_1^2 \tag{8.6}$$

on which $\dot{f}_1$ vanishes, so that Eq. (8.5) reads

$$\operatorname{sgn} \dot{S} = -\operatorname{sgn}[\dot{g}_2(132 f_1^3 - 126 f_1^2 + 54 f_1 - 9 - 48 f_1^4)] \tag{8.7}$$

$\dot{g}_2$ can be expressed with Eqs. (4.2) and (8.6) as

$$\dot{g}_2 = \tfrac{8}{3}(f_1 - \tfrac{3}{4})(f_1 - \tfrac{1}{2})(f_1 - 1) \tag{8.8}$$

So we have a polynomial of seventh degree in f_1 for the entropy production. Calculating its sign, we find

$$\operatorname{sgn} \dot{S} = \begin{cases} -1 & \text{for} \quad 3/4 < f_1 < 1 \\ 0 & \text{for} \quad f_1 = 1/2, 3/4, 1 \\ 1 & \text{for} \quad 0 < f_1 < 3/4, \quad f_1 \neq 1/2 \end{cases} \tag{8.9}$$

All trajectories that start within the s-allowed and the p-allowed region reach the equilibrium fixed point (1/2, 0), i.e., run within its basin. We confirmed this also by actual numerical calculations.

9. RELAXATION TIMES OF THE MCKEAN MODEL

It is usually assumed that the truncation of the BBGKY hierarchy for a general many-body system near the equilibrium is justified, because the higher correlations decay faster than the one-particle distribution function toward equilibrium. In the McKean model this assumption has been proven for infinite particle number N.[6] In order to investigate finite N, we linearize Eqs. (4.2)–(4.6) about the only stable fixed point (1/2, 0), which occurs on every level of truncation. We set therefore

$$g_m = \varepsilon_m \qquad \text{and} \qquad f_1 = \tfrac{1}{2} + \varepsilon_1, \qquad m \geqslant 2$$

Since we expand the correlations about zero, all terms of the cluster expansion[8,9] with more than one correlation vanish, and only products of the ε_m with 1/2 remain. The expansion can be given as a closed formula, which we call the linear cluster expansion:

$$f_m = \sum_{i=0}^{m} \binom{m}{i} \frac{1}{2^i} \varepsilon_{m-i}, \qquad \varepsilon_0 := 1 \tag{9.1}$$

We show in the Appendix that we obtain a new linear hierarchy for the reduced correlations ε_m by substituting Eq. (9.1) into the BBGKY hierarchy (3.7) and, by induction,

$$\dot{\varepsilon}_m = \frac{m(m-1)}{2N}\varepsilon_{m-1} + \left(\frac{m}{N} - m\right)\varepsilon_m + \left(2m - \frac{2m^2}{N}\right)\varepsilon_{m+1} \tag{9.2}$$

We solve these equations with $\varepsilon_m = \exp \lambda_m t$, which yields a secular equation for the eigenvalues λ_m that we expand to first order in terms of $1/N$. The rank of the eigenvalue equations is the level of truncation n. We obtain for the eigenvalues $\lambda_m^{(n)}$,

$$\begin{aligned} \lambda_1^{(2)} &= 3/N - 1 \\ \lambda_2^{(2)} &= -2 \end{aligned} \tag{9.3}$$

for $n = 2$ and

$$\begin{aligned} \lambda_1^{(3)} &= 3/N - 1 \\ \lambda_2^{(3)} &= 12/N - 2 \\ \lambda_3^{(3)} &= -9/N - 3 \end{aligned}$$

for $n = 3$.

Increasing the truncation level changes the second eigenvalue λ_2 and, of course, yields a third one λ_3. This regularity occurs at every level of truncation. We show in the Appendix that the general formula for the eigenvalues is

$$\lambda_m^{(n)} = m\left(\frac{3m}{N} - 1\right) \tag{9.4}$$

for $1 \leqslant m \leqslant n-1$ and

$$\lambda_n^{(n)} = -n - \frac{n^3}{N} + \frac{2n^2}{N} \tag{9.5}$$

for $m = n$.

These formulas indicate an asymptotic convergence in the sense that truncation at a given level n introduces errors which decrease as $N \to \infty$. For the relaxation times we must take a look at the complete solution of Eq. (9.2),

$$\underline{\varepsilon}^{(n)} = \sum_{m=1}^{n} c_m^{(n)} \exp(\lambda_m^{(n)} t)\, \underline{x}_m^{(n)} \tag{9.6}$$

where the $c_m^{(n)}$ are expansion coefficients determined by the initial state and $\underline{x}_m^{(n)}$ are the eigenvectors. With the help of Eqs. (9.4), (9.5), one finds by inspection of Eq. (9.2) that the first m components of $x_m^{(n)}$ are $O(1)$, while the remaining are $O(N^{-1})$. Writing out Eq. (9.6) and dropping the superscript n for simplicity, one has therefore

$$\begin{pmatrix} \varepsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_m \end{pmatrix} = c_1 c^{\lambda_1 t} \begin{pmatrix} \alpha_1 \\ \alpha_2/N \\ \cdot \\ \cdot \\ \alpha_m/N \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3/N \\ \vdots \\ \beta_m/N \end{pmatrix} + \cdots + c_m e^{\lambda_m t} \begin{pmatrix} \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_m \end{pmatrix} \tag{9.7}$$

with all α_m, $\beta_m,\ldots, \gamma_m$ being $O(1)$.

We see *that all* eigenmodes contribute to each correlation ε_m, and the *first* $m-1$ modes of the mth correlation are weighted with a factor $1/N$. For finite N it is therefore not generally true that higher correlations decay faster than the lower ones, as there remain slow components with a small amplitude.

Finally, we note that the linearized hierarchy of correlations (9.2) is connected to the original BBGKY hierarchy (3.7) by the linear cluster expansion (9.1) and other linear operations. In the limit $N \to \infty$ the eigenvalues of the BBGKY hierarchy for the distribution functions are therefore also given by Eq. (9.4).

10. CONCLUSIONS

In this paper we addressed some problems associated with the truncation of evolution equations for reduced densities. We showed that the thermodynamic limit of the kinetic theory has to be taken carefully if one wants to study the time evolution of the entropy including two-particle correlations. In order to obtain a nonvanishing entropy production, model assumptions for the three-particle correlations are needed which must preserve the probability interpretation of the theory for all times. As an example, we used the McKean model with a finite number of particles. We derived the complete BBGKY hierarchy of the distribution functions $\{f_m\}$ for this case, found a one-dimensional equilibrium subspace, and described the boundaries of the p-allowed region in which a probabilistic interpretation of the theory is possible. The correlation functions $\{f_1, g_m \geqslant 2\}$ were introduced with the help of the cluster expansion in order to study the effects of truncation on the nth level by setting $g_{m>n} \equiv 0$. As a consequence of the truncation, there appear saddle points which may lie within the p-allowed region of the $\{f_1, g_{2 \leq m \leq n}\}$ space. There exist therefore trajectories which leave this region during the time evolution of the system. In order to identify the physically allowed basin of the attractive fixed point $(f_1 = 1/2, g_m = 0)$, we study the entropy production of the system. We linearized the hierarchy of correlations about this fixed point, studied truncation effects, and showed in particular that the m-particle correlations g_m include slowly decaying components of amplitude $O(1/N)$. We hope that these results might stimulate similar investigations for more general models.

APPENDIX A

We want to show by induction that Eq. (9.2),

$$\dot{\varepsilon}_m = \frac{m(m-1)}{2N} \varepsilon_{m-1} + \left(\frac{m}{N} - m\right) \varepsilon_m + \left(2m - \frac{2m^2}{N}\right) \varepsilon_{m+1}$$

can be obtained from the BBGKY hierarchy (3.7),

$$\dot{f}_m = \left(\frac{m^2}{N} - m\right) (3f_m - f_{m-1} - 2f_{m+1})$$

by using a linear cluster expansion (9.1):

$$f_m = \left(\frac{1}{2}\right)^m + \sum_{i=0}^{m-1} \binom{m}{i} \frac{1}{2^i} \varepsilon_{m-i}$$

To obtain the first relation we set $m = 1$:

$$\begin{aligned} \dot{\varepsilon}_1 &= \left(\frac{1}{N} - 1\right) (3f_1 - 1 - 2f_2) \\ &= \left(\frac{1}{N} - 1\right)\left(\frac{3}{2} + 3\varepsilon_1 - 1 - \frac{1}{2} - 2\varepsilon_1 - 2\varepsilon_2\right) \\ &= \left(\frac{1}{N} - 1\right) (\varepsilon_1 - 2\varepsilon_2) \end{aligned}$$

Next we calculate the mth equation with the induction assumption that $\dot{\varepsilon}_{m-i}\,(i > 0)$ is of the form (9.2). First we eliminate all distributions by substituting them with the help of Eq. (9.1),

$$\dot{\varepsilon}_m = \left(\frac{m^2}{N} - m\right)(3f_m - f_{m-1} - 2f_{m+1}) - \sum_{i=0}^{m-1} \binom{m}{i} \frac{1}{2^i} \dot{\varepsilon}_{m-i}$$

$$= \left(\frac{m^2}{N} - m\right)\left[\frac{3}{2^m} + \frac{3m}{2^{m-1}}\varepsilon_1 + \sum_{i=0}^{m-2} \binom{m}{i}\right.$$

$$\times \frac{3}{2^i}\varepsilon_{m-i} + 3\varepsilon_m - \frac{1}{2^{m-1}} - \frac{m-1}{2^{m-1}}\varepsilon_1$$

$$- \sum_{i=1}^{m-3} \binom{m-1}{i} \frac{1}{2^i}\varepsilon_{m-i-1} - \varepsilon_{m-1} - \frac{2}{2^{m+1}}$$

$$\left. - \frac{2m+2}{2^m}\varepsilon_1 - \sum_{i=1}^{m-1} \binom{m+1}{i} \frac{2}{2^i}\varepsilon_{m-i+1} - 2\varepsilon_{m+1}\right]$$

$$- \frac{m}{2^{m-1}}\left(\frac{1}{N} - 1\right)(\varepsilon_1 - 2\varepsilon_2) - \sum_{i=1}^{m-2} \binom{m}{i}$$

$$\times \frac{1}{2^i}\left[\frac{i + m^2 + i^2 + m(-2i-1)}{2N}\varepsilon_{m-i-1} + \left(\frac{m-i}{N} - m + i\right)\right.$$

$$\left.\times \varepsilon_{m-i} + \varepsilon_{m+1-i}\left(2m - 2i - \frac{2m^2 + 2i^2 - 4mi}{N}\right)\right]$$

In the next step we change the summation indices and add all constant terms,

$$\dot{\varepsilon}_m = \left(\frac{m^2}{N} - m\right)\left\{3\varepsilon_m - \varepsilon_{m-1} - 2\varepsilon_{m+1} + \sum_{i=2}^{m-2} \varepsilon_{m-i}\left[\binom{m}{i}\frac{3}{2^i}\right.\right.$$

$$\left. - \binom{m-1}{i-1}\frac{1}{2^{i-1}} - \binom{m+1}{i+1}\frac{1}{2^i}\right] + m\frac{3}{2}\varepsilon_{m-1} - (m+1)\varepsilon_m$$

$$\left. - \frac{m^2+m}{4}\varepsilon_{m-1}\right\} + \frac{m}{2^{m-1}}2\varepsilon_2\left(\frac{1}{N} - 1\right) - \sum_{i=2}^{m-3} \varepsilon_{m-i}$$

$$\times \left[\binom{m}{i}\frac{1}{2^i}\left(\frac{m-i}{N} - m + i\right) + \binom{m}{i-1}\frac{1}{2^{i-1}}\frac{i^2 - i - 2im + m + m^2}{2N}\right.$$

$$\left. + \binom{m}{i+1}\frac{1}{2^{i+1}}\left(2m - 2i - 2 - \frac{2m^2 + 2i^2 + 4i + 2 - 4mi - 4m}{N}\right)\right]$$

$$- \frac{m}{2}\left(\frac{m-1}{N} - m + 1\right)\varepsilon_{m-1} - \frac{m}{2}\left(2m - 2 - \frac{2m^2 + 2 - 4m}{N}\varepsilon_m\right)$$

$$- \frac{m(m-1)}{8}\left(2m - 4 - \frac{2m^2 - 8m + 8}{N}\right)\varepsilon_{m-1} - \frac{m(m-1)}{2^{m-1}}\left(\frac{2}{N} - 2\right)\varepsilon_2$$

$$- \frac{m(m-1)(m-2)}{N2^{m-2}}\varepsilon_2$$

Now we collect all correlations that stand out of the sums and the factor $1/2^i$ out of the brackets,

$$= \left(\frac{m^2}{N} - m\right)\Bigg((-m+2)\,\varepsilon_m + \left(\frac{5}{4}m - \frac{m^2}{4} - 1\right)\varepsilon_{m-1} - 2\varepsilon_{m+1}$$

$$+ \sum_{i=2}^{m-3} \varepsilon_{m-i}\frac{1}{2^i}\left[\binom{m-1}{i}2 - \binom{m}{i+1}\right]\Bigg) - \sum_{i=2}^{m-3}\varepsilon_{m-i}\frac{1}{2^i}$$

$$\times\Bigg[\binom{m}{i}\left(\frac{m-i}{N} - m + i\right) + \binom{m}{i-1}\frac{i^2 - i - 2im + m + m^2}{N}$$

$$+ \binom{m}{i+1}\left(m - i - 1 - \frac{m^2 + i^2 + 2i + 1 - 2mi - 2m}{N}\right)\Bigg]$$

$$-\frac{m}{2}\left(\frac{m-1}{N} - m + 1\right)\varepsilon_{m-1} - \frac{m}{2}\left(2m - 2 - \frac{2m^2 + 2 - 4m}{N}\right)\varepsilon_m$$

$$-\frac{m(m-1)}{8}\left(2m - 4 - \frac{2m^2 - 8m + 8}{N}\right)\varepsilon_{m-1}$$

In order to combine the sums, we have to rearrange the binomial coefficients

$$\dot{\varepsilon}_m = \left(\frac{m^2}{N} - m\right)\left[(2-m)\,\varepsilon_m + \left(\frac{5}{4}m - \frac{m^2}{4} - 1\right)\varepsilon_{m-1} - 2\varepsilon_{m+1}\right]$$

$$+ \sum_{i=2}^{m-3}\varepsilon_{m-i}\frac{1}{2^i}\binom{m}{i}\Bigg[\frac{2m^2 - 2im}{N} - 2m + 2im - \frac{m^3 - im^2}{(i+1)\,N}$$

$$-\frac{(m-i)(m-i-1)}{i+1} + \frac{m^2 - im}{i+1} - \left(\frac{m-i}{N} - m + 1\right)$$

$$-\frac{i(m-i)^2 + im - i^2}{(m-i+1)\,N} + \frac{(m-i)^3 + (m-i)(2i+1-2m)}{(i+1)\,N}\Bigg]$$

$$-\varepsilon_m\left(m^2 - m - \frac{m^3 + m - 2m^2}{N}\right) - \varepsilon_{m-1}\Bigg[\frac{m^2 - m}{8}$$

$$\times\left(2m - 4 - \frac{2m^2 - 8m + 8}{N}\right) + \frac{m^2 - m}{2N} - \frac{m^2}{2} + \frac{m}{2}\Bigg]$$

Each term in the sum vanishes. Combining the remaining terms shows

$$\dot{\varepsilon}_m = \frac{m(m-1)}{N}\varepsilon_{m-1} + \left(\frac{m}{N} - m\right)\varepsilon_m + \left(2m - \frac{2m^2}{N}\right)\varepsilon_{m+1}$$

QED

APPENDIX B

We calculate the eigenvalues $\lambda_m^{(n)}$ of the determinant $|A^{(n)}|$ where the matrix $A^{(n)}$ consists of the coefficients of Eq. (9.2) minus $\lambda^{(n)}E$, with the unit matrix E. We expand the determinant $|A^{(n)}|$ with respect to its last two rows

$$|A^{(n)}| = \left(\frac{n}{N} - n - \lambda^{(n)}\right) |A^{(n-1)}| - \frac{n(n-1)}{2N}\left[2(n-1) - \frac{2(n-1)^2}{N}\right] |A^{(n-2)}|$$

The induction assumption is now that the eigenvalues of the subdeterminants $|A^{(n-1)}|$ and $|A^{(n-2)}|$ differ only in the $1/N$ correction of the eigenvalues $\lambda_{n-2}^{(n-1)}$ and $\lambda_{n-2}^{(n-2)}$. Factorizing the subdeterminants, respectively, one obtains

$$\begin{aligned} |A^{(n)}| = &\left[\frac{n}{N} - n - \lambda^{(n)}\right]\left[\prod_{m=1}^{n-3} (\lambda_m^{(n-1)} - \lambda^{(n)})\right] (\lambda_{n-2}^{(n-1)} - \lambda^{(n)})(\lambda_{n-1}^{(n-1)} - \lambda^{(n)}) \\ &- \frac{n(n-1)}{N}\left[\prod_{m=1}^{n-3} (\lambda_m^{(n-2)} - \lambda^{(n)})\right] (\lambda_{n-2}^{(n-2)} - \lambda^{(n)}) \end{aligned}$$

Here the $1/N^2$ factor of the second term has already been neglected. The critical point now is that the difference between $\lambda_{n-2}^{(n-1)}$ and $\lambda_{n-2}^{(n-2)}$ can be neglected in the expansion up to first order of $1/N$. Therefore we can write

$$|A^{(n)}| = \left[\prod_{m=1}^{n-2} (\lambda_m^{(n-1)} - \lambda^{(n)}\right]\left[\left(\frac{n}{N} - n - \lambda^{(n)}\right) (\lambda_{n-1}^{(n-1)} - \lambda^{(n)}) - \frac{n(n-1)^2}{N}\right]$$

For the last factor we obtain the desired two eigenvalues $\lambda_{n-1}^{(n)}$ and $\lambda_n^{(n)}$.

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